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THE LOSS OF STABILITY OF SYMMETRICAL EQUILIBRIUM POSITIONS†

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Systems of differential equations possessing a finite (or compact) symmetry group and depending on one parameter are considered. The nature of the loss of stability of equilibrium positions is investigated in cases when, owing to symmetry, the linearized problem has multiple eigenvalues. Conditions are presented that determine whether the loss of stability when the parameter is varied is soft or hard, for double eigenvalues λ - zero or pure imaginary. Cases of triple zero eigenvalues λ, corresponding to tetrahedral (or cubic) symmetry, are considered. © 1999 Elsevier Science Ltd. All rights reserved.

Let $\mathbf{r}' = \mathbf{f}(\mathbf{r}, \alpha)$ be a system of differential equations which depends on one (scalar) parameter. Suppose that in some range of variation of the parameter the system has an asymptotically stable equilibrium position $\mathbf{c}(\alpha)$. For simplicity, we will assume that the interval is given by the inequalities $\alpha_1 < \alpha < 0$. As is well known, in a typical family of equations of general form there are only two scenarios for the equilibrium to lose its stability as the parameter is varied: (1) at a critical parameter value $\alpha = 0$ one branch of $\mathbf{c}(\alpha)$ merges with another branch $\tilde{\mathbf{c}}(a)$, and as the parameter continues to vary they "annihilate" (more precisely: for small $\alpha > 0$ in some domain U containing $\mathbf{c}(0)$, there are no equilibrium positions); (2) as α varies, the equilibrium position $\mathbf{c}(\alpha)$ evolves smoothly; it becomes unstable when $\alpha > 0$ and a limit cycle of small amplitude forms around it (in this situation the stability loss is soft if the cycle is born when $\alpha > 0$ and hard otherwise; see [1, Section 33]). The first case occurs when one eigenvalue λ of the linearization operator $L = \mathbf{f}_\mathbf{r}'(\mathbf{c}(0), 0)$ vanishes (in that case one need not speak of "loss of stability of the equilibrium", for the equilibrium itself disappears). The second case occurs when a pair of complex conjugate eigenvalues crosses the imaginary axis.

The general rules are not applicable if the system possesses some kind of symmetry. A frequently encountered case is a system whose right-hand side is an odd function of $\mathbf{r}$. Here $\mathbf{r} = \mathbf{0}$ is an equilibrium position for all α. When it happens that $\lambda = 0$, a pair of equilibria $\mathbf{c}^{\pm}(\alpha)$ split off this equilibrium—a "pitchfork" bifurcation occurs. Here, again, the stability loss may be either soft or hard.

This example, however, lacks an important feature of symmetric systems: the presence of multiple eigenvalues λ at all parameter values. Multiplicity of this sort may occur if the symmetry group is *non-commutative*, in which case the events accompanying the stability loss may be extremely complicated. For example, "chaotic" low-amplitude oscillations may split off the equilibrium if, as the parameter passes through the critical value, a quadruple zero eigenvalue λ [2] or a triple pair of pure imaginary λ's [3] appear.

As in general systems, the nature of the loss of stability of an equilibrium position as the parameter varies is determined by what happens "at the critical moment." If asymptotic stability is still preserved at $\alpha = 0$, the stability loss when $\alpha > 0$ is soft. But if the equilibrium is unstable when $\alpha = 0$, the stability loss is hard [4, Section 44; 5, Chapter 1, Section 5].

In this paper we will present stability criteria when $\alpha = 0$ for cases of double critical (zero or pure imaginary) eigenvalues λ and triple zero λ's.‡ Basically, we will be considering finite symmetry groups: in the case of continuous symmetry groups (and isolated equilibrium) the problems are usually simplified. As in the case when there is no symmetry, it is much easier to investigate the stability problem when $\alpha = 0$ than to solve the bifurcation problem—to delineate the local phase portrait for small α. In many cases here, an answer is obtained although the full bifurcation picture is not known. This is precisely the case for half of the cases with double imaginary λ (Section 3) and for the problems in Sections 5 and 7.

In order to implement the stability criteria given below one has to find the factors appearing in them. This is a separate non-trivial problem (see [6]) requiring computer simulation (see [7, Chapter 10]).

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‡The details, including part of the proofs, will be omitted here. See SHNOL', E. E., The loss of stability of equilibrium positions in symmetrical systems of differential equations. Preprint. Pushchino Scientific Centre of the Russian Academy of Sciences, 1998.

1. STATEMENT OF THE PROBLEM AND ADDITIONAL INFORMATION

Let (1.1) be a one-parameter family of differential equations which is invariant under a finite (or compact) group Γ of linear transformations

$$d\mathbf{r}/dt = \mathbf{f}(\mathbf{r}, \alpha), \quad \mathbf{r}, \mathbf{f} \in \mathbf{R}^n; \quad \mathbf{f}(\mathbf{0}, \alpha) \equiv \mathbf{0} \tag{1.1}$$

$$\mathbf{f}(g\mathbf{r}, \alpha) = g\mathbf{f}(\mathbf{r}, \alpha), \quad g \in \Gamma \tag{1.2}$$

We may assume without loss of generality that the elements g of Γ are orthogonal transformations in the sense of some scalar product. For simplicity, we will assume that $\mathbf{r} = \mathbf{0}$ is an equilibrium position for all values of the parameter α (possessing maximum symmetry). The general case, in which the equilibrium position $\mathbf{c}(\alpha)$ is invariant only under some subgroup of Γ, is easily reduced to this special case if Γ is a finite group. It is assumed that for $\alpha_1 < \alpha < 0$ all the eigenvalues λ of the linearization operator $L_\alpha = \mathbf{f}'_{\mathbf{r}}(\mathbf{0}, \alpha)$, lie in the left half-plane, but when $\alpha = 0$ some λ's are on the imaginary axis.

Problem. It is required to determine the nature (soft or hard) of the loss of stability of the equilibrium $\mathbf{r} = \mathbf{0}$ for a given group Γ. Or, what is the same: it is required to determine whether the equilibrium at the critical value $\alpha = 0$ is stable.

The phase space may be expressed, not necessarily uniquely, as a direct sum of orthogonal subspaces invariant under the group Γ and irreducible (not containing proper invariant subspaces)

$$\mathbf{R}^n = E_1 \oplus \ldots \oplus E_s \tag{1.3}$$

When considering the eigenvalues of the operator L_α one has to work in the complex domain, first extending the action of the linear transformations g in a natural way. Each subspace E_k then become a complex subspace $\tilde{E}_k$. If E_k is of odd dimension, then $\tilde{E}_k$ will again be irreducible. If E_k is even-dimensional, there are two possibilities: (1) $\tilde{E}_k$ splits into the sum of two invariant subspaces of half the dimension; (2) $\tilde{E}_k$ is irreducible. In the latter case we will call E_k an *absolutely irreducible subspace*.

Let us assume that the operator L_0 has a ν-fold eigenvalue $\lambda = 0$. The corresponding eigensubspace Ei will be invariant under the action of the symmetry group Γ. The following situation is "typical", that is, is preserved under an arbitrarily small perturbation of the initial family of equations that does not destroy its symmetry (or corresponds to codimension 1 in the class of symmetric equations): E consists of eigenvectors and is one of the absolutely irreducible subspaces E_k. For ν-fold imaginary λ, the subspace E (of dimension 2ν) may be either reducible or irreducible (see [8, Chapter 16; 9] and the Appendix).

Example. Let m identical two-dimensional systems be combined into a ring by linear connections of the "diffusion" type. The full system of $2m$ equations is

$$\mathbf{q}'_k = \mathbf{h}(\mathbf{q}_k) + D(\mathbf{q}_{k+1} - \mathbf{q}_k) + D(\mathbf{q}_{k-1} - \mathbf{q}_k), \quad k = 0, 1, \ldots, m-1 (\mathrm{mod}\, m) \tag{1.4}$$

where $\mathbf{q}$ and $\mathbf{h}$ are two-dimensional and D is a diagonal matrix (positive d_1 and d_2 on the diagonal). If the two-dimensional system $\mathbf{q}' = \mathbf{h}(\mathbf{q})$ has no "internal" symmetry, the group Γ for (1.4) is isomorphic to the complete symmetry group of a regular m-gon. There is a two-dimensional subgroup invariant under the action of Γ, call it $E^{(0)}$; this subgroup is defined by the conditions $\mathbf{q}_0 = \ldots = \mathbf{q}_{m-1}$; any straight line in $E^{(0)}$ is also invariant. If m is odd, there are $(m-1)/2$ further four-dimensional invariant subspaces $E^{(l)}$; in $E^{(l)}$

$$\mathbf{q}_k = \mathbf{a} \cos(k\theta_l) + \mathbf{b} \sin(k\theta_l), \quad \theta_l = 2\pi l/m$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary two-dimensional vectors. Each subspace $E^{(l)}$ splits (possibly in more than one way) into the sum of two-dimensional subspaces which are invariant under the group Γ. Thus, the decomposition (1.3) contains two one-dimensional and $m-1$ two-dimensional terms. (If m is even, the subspace $E^{(m/2)}$ is two-dimensional and (1.3) contains four one-dimensional terms.)

System (1.4) has symmetric equilibrium positions: all $\mathbf{q}_k = \mathbf{q}_*$, $\mathbf{h}(\mathbf{q}_*) = \mathbf{0}$. If $d_1 \neq d_2$ and any parameter is varied, one of these equilibrium positions may lose its stability with a loss of symmetry—that is, it will leave the subspace $E^{(0)}$ (which is invariant for Eqs (1.4)). When that happens, double zero or pure imaginary eigenvalues appear and discrete analogues of "dissipative structure" or waves are created.

Remark. In this comparatively simple example, all bifurcation events are known (see, in particular, [8, Chapter 18]).

There is a general theorem that enables one, when considering bifurcations, to study a system of differential equations whose dimension is the number of eigenvalues on the imaginary axis. This is the

Centre Manifold or Neutral Manifold Theorem ([1, Section 32; [10, Chapter 2). An analogous result was established previously for the stability problem [11]. In what follows we will consider precisely such systems, which have the minimum possible dimension for a given group Γ. In such situations not all the group Γ "works", but only its restriction G to a subspace E (or, what is the same, to the centre manifold).

2. DOUBLE ZERO λ

Suppose the two-dimensional system $\mathbf{r}' = \mathbf{f}(\mathbf{r})$ is invariant under a group of orthogonal transformations G with absolutely irreducible action in $\mathbf{R}^2$ (see Section 1). There are two possibilities: (1) G is the complete group of orthogonal transformations $\mathbf{O}(2)$, (2) G is $\mathbf{D}_m$—the complete symmetry group of a regular m-gon.

By assumption, $L = \mathbf{f}'(\mathbf{0}) = \mathbf{0}$, that is, there are no linear terms in $\mathbf{f}(\mathbf{r})$. The problem of the stability of the equilibrium $\mathbf{r} = \mathbf{0}$ in the first case is trivial. Here the system has the form

$$x' = axr^2 + O(r^4), \quad y' = ayr^2 + O(r^4)$$

and stability depends on the sign of a.

For the group $\mathbf{D}_m$, let us introduce Cartesian coordinates x and y so that the x axis is an axis of symmetry of the m-gon; note that each axis of symmetry is an invariant straight line for the system. Then systems with symmetry $\mathbf{D}_m$ in the complex coordinate $\xi + x + iy$ will have the following form (a and b are real)

$$\begin{aligned} m = 3:&\ \ \zeta' = a\bar{\zeta}^2 + O(r^3), \quad r^2 = |\zeta|^2 = x^2 + y^2 \\ m = 4:&\ \ \zeta' = a\zeta|\zeta|^2 + b\bar{\zeta}^3 + O(r^4) \\ m \geqslant 5:&\ \ \zeta' = a\zeta|\zeta|^2 + O(r^4) \end{aligned} \tag{2.1}$$

Remark. Such systems arise when investigating bifurcations of limit cycles in general, non-symmetric systems [1, Section 35]. There, however, the coefficients are complex.

For $m = 3$ and $m \geqslant 5$ the answer to the question of stability is very simple. If $m = 3$, we have on the x axis an equation $x' = ax^2 + O(x^3)$: the equilibrium position $x = y = 0$ is unstable if $a \neq 0$. If $m \geqslant 5$, one has asymptotic stability when $a < 0$ and instability when $a > 0$: $\Lambda = x^2 + y^2$ is a Lyapunov function.

Proposition 2.1. The equilibrium position $x = y = 0$ of system (2.1) with symmetry $\mathbf{D}_4$ is asymptotically stable provided that $a < 0$, $|a| > |b|$, or

$$a + b < 0, a - b < 0 \tag{2.2}$$

The equilibrium position is unstable if $a > 0$ or $|a| < |b|$.

Proof. Set $\Lambda = |\zeta|^2$. Along trajectories of system (2.1), $\Lambda' < 2ar^4 + 2|b|r^4 + O(r^5)$. If (2.2) is true, then $\Lambda' < 0$ for $0 < |\zeta| < r_*$, and the equilibrium position $\zeta = 0$ is asymptotically stable. On the x axis we have $x' = 2(a - b)x^3 + O(x^4)$, and the instability for $a + b > 0$ is obvious. On the (invariant) line $x = y$ we have $x' = 2(a - b)x^3 + O(x^4)$, and instability for $a - b > 0$ is also obvious.

3. DOUBLE IMAGINARY I

A double pair of pure imaginary eigenvalues arises when one parameter is varied in two ways (see the Appendix). The bifurcation pictures in these situations are very different (and known only partially‡), but for the problem of stability at the critical moment this difference is not significant. In both cases the system of four equations on the centre manifold at $\alpha = 0$, after reduction to normal form up to the third order inclusive, has the following form

$$\begin{aligned} \zeta_1' &= i\omega\zeta_1 + \zeta_1(A_1r_1^2 + A_2r_2^2) + O(r^4), \quad r_k = |\zeta_k| \\ \zeta_2' &= i\omega\zeta_2 + \zeta_2(A_2r_1^2 + A_1r_2^2) + O(r^4), \quad r^2 = r_1^2 + r_2^2 \end{aligned} \tag{3.1}$$

†See SHNOL' E. E. and NIKOLAYEV Ye V., Bifurcations of equilibrium positions in systems of differential equations with a finite symmetry group. Preprint. Pushchino Scientific Centre of the Russian Academy of Sciences, 1997.

where ζ_1 and ζ_2 are complex variables and A_1 and A_2 are complex coefficients; in the first case, as a rule, $A_1 \neq A_2$; in the second, on the contrary, $A_1 = A_2$ always. System (3.1) has the same appearance as in the problem of stability with two different pairs of imaginary eigenvalues ($\pm i\omega_1$, $\pm i\omega_2$). More precisely, it is obtained from the general notation by putting $\omega_1 = \omega_2$ and choosing the coefficients of the cubic terms in a special way.

The stability criterion for this case (due to G. V. Kamenkov; see [12] and [13, Section 1.3]) yields: the equilibrium position (0, 0) of system (3.1) is asymptotically stable if the following two conditions are satisfied: (1) $a_1 < 0$; (2) $a_1 + a_2 < 0$ ($a_k = \mathrm{Re}\,(A_k)$); the equilibrium position is unstable if $a_1 > 0$ or $a_1 + a_2 > 0$; in the second case, when $a_1 = a_2 = a$, stability depends on the sign of a.

4. TRIPLE ZERO λ

Here G is a subgroup of the group $\mathbf{O}(3)$ of orthogonal transformations of three-dimensional space with irreducible action, that is, it has no invariant subspaces of dimensions 1 and 2. Let G_+ denote the subgroup of all proper orthogonal transformations in G; each such transformation is a rotation about some axis l.

The right-hand sides of the system form a vector field $\mathbf{f}$ which is invariant under G in the sense of (1.2). If the straight line l is a rotational axis of symmetry, then $\mathbf{f}$ is parallel to l; this straight line is invariant for the equations.

We have the following possibilities for the group G_+, in which G acts irreducibly [(14 Section, 13; [15], Section 93]): (1) G_+ is the whole group of proper orthogonal transformations $\mathbf{SO}(3)$; (2) G_+ is the group T of rotations of a tetrahedron (12 elements); (3) G_+ is the group O of rotational symmetries of a cube (or a regular octahedron), which contains 24 elements; (4) G_+ is the group Y of rotations of a regular dodecahedron (or icosahedron).

Case 1 is trivial. Later we will consider Case 2 in detail and present the results for Cases 3 and 4.

5. TETRAHEDRAL SYMMETRY (G_+ IS THE GROUP T OF ROTATIONS OF A TETRAHEDRON)

Let x, y, z be Cartesian coordinates; we write the system of three equations as

$$x' = u(\mathbf{r}, \alpha),\ y' = v(\mathbf{r}, \alpha),\ z' = w(\mathbf{r}, \alpha) \tag{5.1}$$

When $\alpha = 0$ there are, by assumption, no linear terms in u, υ and w. By (1.2), under transformations of G the vector field $\mathbf{f} = (u, \upsilon, w)$ transforms like $\mathbf{r} = (x, y, z)$. The same holds for any homogeneous component of the field $\mathbf{f}_k = (u_k, \upsilon_k, w_k)$ ($\mathbf{f}_k$ is the set of terms of degree k in the Taylor expansion of $\mathbf{f}$ in powers of $\mathbf{r}$).

Let us position a regular tetrahedron in such a way that its axes of symmetry of second order (passing through the mid-points of opposite edges) coincide with the coordinate axes. Then the four axes of third-order symmetry (passing through the vertices) coincide with the bisectors of the coordinate octants: $x = \pm y = \pm z$. We shall say that the vector field is invariant under the group of linear transformations if it satisfies relations of type (1.2) (such vector fields are also known as "equi-variant" fields).

Lemma 5.1. A quadratic vector field $\mathbf{f}_2$ which is invariant under the group T (with the representation described above) has the form

$$u_2 = ayz,\ u_2 = azx,\ w_2 = axy$$

Proof. In a rotation about the x axis through 180°, $x \to x, y \to -y, z \to -z$, and necessarily also $u \to u$. Hence $c = 0$ and $P_2(y, z) = ayz$. The form of υ_2 and w_2 is established in an analogous way. That the coefficients are identical follows from the fact that rotation through 120° about the axis $x = y = z$ takes $u \to \upsilon$ and $\upsilon \to w$.

Remark. A quadratic field $\mathbf{f}_2$ is also invariant under the complete symmetry group T_d of the tetrahedron (including reflections).

By Lemma 5.1, the system of equations has the following form when $\alpha = 0$

$$x' = ayz + \ldots,\ y' = azx + \ldots,\ z' = axy + \ldots$$

where the dots stand for terms $O(r^3)$. The equilibrium position (0, 0, 0) is unstable at $a \neq 0$. Indeed, the straight line $l: x = y = z$ is invariant for the system (see Section 4). On l the system reduces to an equation $x' = ax^2 + O(|x|^3)$, and instability is obvious. The foregoing implies the following.

Proposition 5.1. Suppose a system $\mathbf{r}' = \mathbf{f}(\mathbf{r}, \alpha)$ of three differential equations is invariant under the rotation group T of a tetrahedron or its complete symmetry group T_d. Then the system has the form

$$\mathbf{r}' = \lambda(\alpha)\mathbf{r} + a(\alpha)\mathbf{s}_2(\mathbf{r}) + O(|\mathbf{r}|^3)$$

Let $\lambda(0) = 0$ and suppose the following non-degeneracy conditions are satisfied: (1) $\lambda'(0) \neq 0$, (2) $a(0) \neq 0$. Then if $\alpha = 0$ the equilibrium position $\mathbf{r} = \mathbf{0}$ experiences a hard stability loss. In suitable variables, $\mathbf{s}_2(\mathbf{r}) = (yz, zx, xy)$.

We will consider one further possibility. When $G_+ = T$, the complete symmetry group G may be obtained by adding a central symmetry (see [13], Sections 93, X). Under such a group G the right-hand sides of the system are odd functions, there are no quadratic terms, and instability at $\alpha = 0$ is no longer the rule.

Lemma 5.2. A vector field $\mathbf{f}_3$ invariant under the group T (with the above representation) has the form

$$u_3 = x(ax^2 + by^2 + cz^2), \ v_3 = y(ay^2 + bz^2 + cx^2) \quad w_3 = z(az^2 + bx^2 + cy^2) \tag{5.2}$$

Proof. We write

$$u_3 = ax^3 + x^2P_1(y, z) + xP_2(y, z) + P_3(y, z)$$

For rotation about the x axis through 180° we have necessarily $u \to u$. Hence $P_1 = P_3 \equiv 0$. For rotation about the y axis through 80° necessarily $u \to -u$. Hence $P_2(y, z) = by^2 + cz^2$. For rotation about the axis $x = y = z$ through 120°, $u \to v$ and $v \to w$, and therefore the corresponding coefficients in formulae (5.2) are identical.

Proposition 5.2. Suppose a system $\mathbf{r} = \mathbf{f}(\mathbf{r})$ of three differential equations is invariant under the group G (generated by the group T and central symmetry). Then

1. In suitable variables, the system has the form $\mathbf{r}' = \mathbf{f}_3(\mathbf{r}) + O(|\mathbf{r}|^4)$, and the cubic terms $\mathbf{f}_3(\mathbf{r})$ are as described in Lemma 5.2.
2. The equilibrium position $\mathbf{r} = \mathbf{0}$ of the system is asymptotically stable provided that

$$a < 0, a + b + c < 0 \tag{5.3}$$

3. The equilibrium position is unstable if $a > 0$ or $a + b + c > 0$.

Proof. 1 follows from the lemma.

2. When inequalities (5.3) are true, $\Lambda = 1/2|\mathbf{r}^2|$ is a Lyapunov function. Indeed, the derivative Λ' along trajectories of the system is

$$a(x^4 + y^4 + z^4) + (b + c)(x^2y^2 + y^2z^2 + z^2x^2) + O(r^5) \ (r = |\mathbf{r}|)$$

If $a < 0$ and $b + c < 0$, then $\Lambda' < 0$ for $0 < r < r_*$.

If $a < 0$ and $b + c \geqslant 0$, then

$$\Lambda' < (a + b + c)(x^4 + y^4 + z^4) + O(r^5)$$

The right-hand side of this inequality is negative for $0 < r < r_*$ if $a + b + c < 0$.

3. The x axis is invariant for the system. On it, $x' = ax^3 + O(x^4)$, and instability when $a > 0$ is obvious. On the (invariant) straight line $x = y = z$ we have $x' = (a + b + c)x^3 + O(x^4)$, and instability when $a + b + c > 0$ is also obvious.

6. CUBIC SYMMETRY (G_+ IS THE GROUP O OF ROTATIONAL SYMMETRIES OF A CUBE)

Choose the Cartesian system of coordinates so that its axe are parallel to the edges of the cube and its origin is the centre of the cube.

Suppose a system of three differential equations is invariant under the rotation group O of the cube or the complete group O_h of symmetries of the cube. Then in the variables indicated the statement obtained by setting $b = c$ in Proposition 5.2 holds.

7. DODECAHEDRAL SYMMETRY

In this case there are no quadratic terms in Eqs (5.1), while the cubic terms have the same form as in spherical symmetry.

Proposition 7.1. Suppose a system of three differential equations is invariant under the group Y of rotations of a dodecahedron. Then

1. the system has the form $x_k' = ax_kr^2 + O(r^4)$ (x_k $(k = 1, 2, 3)$ are Cartesian coordinates);
2. the equilibrium position (0, 0, 0) is asymptotically stable when $a < 0$ and unstable when $a > 0$.

APPENDIX
(DOUBLE IMAGINARY λ IN PROBLEMS OF CODIMENSION 1)

Let $\mathbf{R}^n = E_1 + \ldots + E_s$ be a decomposition of the phase space into irreducible components (see Section 1). Let Γ_k denote the restriction of the group Γ to the invariant subspace E_k. The correspondence $\Gamma \to \Gamma_k$ yields a (linear) irreducible representation of the group Γ in the space E_k. Double imaginary eigenvalues of the operator L_α may arise (in a non-removable way) when a single parameter is varied, in two cases [8, 9]:

1. A four-dimensional subspace E_k exists for which the representation $\Gamma \to \Gamma_k$ is not absolutely irreducible (it splits upon complexification).
2. Two two-dimensional subspaces E_i and E_j exist for which the representations are absolutely irreducible and isomorphic.

In the first case the four-dimensional eigensubspace E corresponding to a double $\lambda = \pm\omega$ is E_k. In the second case E is the direct sum of E_i and E_j. Whether necessarily $A_1 = A_2$ in formula (3.1), depends on the group G—the restriction of Γ to the subspace E (see the paper referred to in the footnote to Section 3).

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